$$\Delta\Omega_n^2 = -\Omega_n^4/M_{1n}\{x_{1n}\}^T[M_{12}-K_{12}K_{22}^{-1}M_{22}] \times [-\Omega_n^2 M_{22}+K_{22}]^{-1}[M_{21}-M_{22}K_{22}^{-1}K_{21}]\{x_{1n}\} \quad (11)$$

where

$$M_{1n} = \{x_{1n}\}^T[M_{11}]\{x_{1n}\}$$

In these equations for $\{x'_{2n}\}$ and $\Delta\Omega_n^2$, it is not permissible to use a power series expansion in $\omega^2$ for the inversion of the matrix unless $\Omega_n^2$ is less than any eigenvalue of the system

$$(-\omega^2 M_{22}+K_{22})\{x_2'\} = 0 \quad (12)$$

Except for the special cases when $M_{22}$ and $K_{22}$ are diagonal or when $(M_{12}-K_{12}K_{22}^{-1}M_{22}) = 0$ (in which case the coupling coefficients vanish), the solution for $\{x'_{2n}\}$ requires inversion of the matrix, $(-\Omega_n^2 M_{22}+K_{22})^{-1}$.

If all of the eigenvalues and eigenvectors of Eq. (12) are known, then the inverse matrix may be written in the well-known eigenfunction expansion form and Eq. (10) becomes

$$\{x'_{2n}\} = \sum_m \frac{\{Y_m\}\{Y_m\}^T\{F_n\}}{M_{2m}\omega_m^2(1-\Omega_n^2/\omega_m^2)} \quad (13)$$

where $\omega_m^2$ and $\{Y_m\}$ are, respectively, the $m$th eigenvalue (assumed to be distinct) and eigenvector of Eq. (12). $M_{2m}$ is given by $M_{2m} = \{Y_m\}^T[M_{22}]\{Y_m\}$ and $F_n$ is the column matrix

$$\{F_n\} = [M_{21}-M_{22}K_{22}^{-1}K_{21}]\{x_{1n}\} \quad (14)$$

Term by term, it can be seen from Eq. (13) that the power series representation of the "resonance" denominator is

$$1/1-\Omega_n^2/\omega_m^2 = 1+\Omega_n^2/\omega_m^2+\Omega_n^4/\omega_m^4+\cdots \quad (15)$$

and this is convergent only if $\Omega_n^2/\omega_m^2 < 1.0$. Thus, the power series representation can be used only if $\Omega_n^2/\omega_m^2$ is less than one. In other cases it may led to "corrections" not only of the wrong magnitude but of the wrong sign. For the lowest eigenvalues, $\Omega_n^2$, it may happen that none of the $\omega_m$ are exceeded so that the power series approximation may be valid. If only one or a few of the $\omega_n^2$'s are exceeded, the corresponding eigenvalues and eigenvectors may be found by iteration and used to form the proper terms in the eigenfunction expansion, Eq. (13), while the terms corresponding to higher eigenvalues of $\omega_n^2$ are approximated by a power series representation similar to that proposed by Kidder. The details of this procedure have been presented in Ref. 8 for a similar eigenvalue problem in aeroelasticity.

Of course the approximation process of perturbation theory described above may be continued formally to higher order terms although this may be a long process if the perturbation series is poorly convergent. Also, instead of calculating the second-order correction to frequency from Eq. (11), the eigenvectors $\{x_{1n}\}$ and $\{x_{2n}\}+\{x'_{2n}\}$ may be used to substitute for $\{x_1\}$ and $\{x_2\}$, respectively, in the Rayleigh quotient corresponding to Eq. (1) to obtain new approximations for the frequencies. The latter procedure will usually be more accurate, at least for the lower frequencies.

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# Reply by Author to A. H. Flax

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THE author wishes to thank A. H. Flax for his comments and for calling attention to the convergence requirement on the series expansion

$$(-\omega^2 M_{22}+K_{22})^{-1} = K_{22}^{-1}+\omega^2 K_{22}^{-1}M_{22}K_{22}^{-1}+\cdots \quad (1)$$

wherein Eq. (1) will always converge for frequencies $\omega$ which are less than the smallest frequency $\bar{\omega}$ obtained from the eigenproblem

$$(-\bar{\omega}^2 M_{22}+K_{22})\{x_2\} = \{0\} \quad (2)$$

However, the author must take exception to Flax's erroneous assertion that the reduction method of Eqs. (5–7) of Ref. 1 was not derived correctly. The point Flax has overlooked is that the reduction method of an eigenproblem is an approximation technique to obtain the *lowest* modes and frequencies of the total structure and avoid the computational difficulties of solving a large eigenproblem. For the reduction method to be properly applied, it is implicitly assumed that the degrees of freedom retained must be those associated with the lowest modes and frequencies of the structure and those eliminated with the higher modes and frequencies. If this is not accomplished, the reduction procedure yields worthless results.

For a large complex structure where the degrees of freedom corresponding to local modes of low frequency have been inadvertently eliminated, it is doubtful that the results of the reduction procedure could be adequately refined, using the perturbation technique presented by Flax, to recover these local modes and their corresponding frequencies. Therefore, when the reduction method is used properly, the solution frequencies of interest will automatically satisfy the convergence requirement a priori; the expansion of Eq. (1) and the simplified back-transformation expression

$$\{x_2\} = -(K_{22}^{-1}+\omega^2 K_{22}^{-1}M_{22}K_{22}^{-1})(-\omega^2 M_{21}+K_{21})\{x_1\} \quad (3)$$

will both be valid approximations.

Usually, the analyst can only rely on his engineering judgment and expertise to make the proper selection as to which degrees of freedom to retain and which to eliminate. But now, by investigating the approximations involved in the series expansion of Eq. (1), there is a definite criterion to be used in making a selection. That is, in order to obtain the lowest frequencies in the reduced problem, eliminate those degrees of freedom which maximize the lowest frequency of the convergence requirement Eq. (2). For a complicated structure it may be difficult to determine how to implement this selection procedure, but, at least, there now is a guideline to follow. It should be pointed out, as shown by Eq. (2), that both the mass (kinetic energy) and the stiffness (potential energy) coefficients must be considered when selecting degrees of freedom to eliminate. To illustrate the reduction method and the back-transformation scheme of Eq. (3) the following example problem is presented.

Figure 1 depicts a prismatic beam simply supported; it is desired to investigate the small lateral vibrations of this beam in the $x$-$y$ plane. The beam is characterized by the parameters $A$, $E$, $I$, $\rho$, and $L$ which are the cross sectional area, modulus of elasticity, moment of inertia, mass density, and half-length of the beam, respectively. The beam has been discretized to three node points with four degrees of freedom, a node at each end, each with a rotational degree of freedom, and a node at midspan with a lateral translational and a rotational degree of freedom. The problem is to be reduced to two degrees of freedom by

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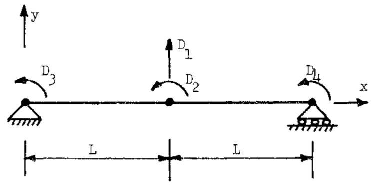


**Fig. 1 Simply supported beam.**

eliminating the two rotational degrees of freedom at the ends of the beam. Using the consistent mass and stiffness matrices for the beam elements[2] and the reduction method[1,3] to eliminate degrees of freedom $D_3$ and $D_4$, yields the eigenvalues

$$\omega_1{}^2 = (105/17)EI/\rho AL^4 \quad \omega_2{}^2 = 157.5\, EI/\rho AL^4 \tag{4}$$

The auxiliary eigenvalue problem, Eq. (2), which establishes convergence of Eq. (1) yields the double root solution

$$\bar{\omega}_1{}^2 = \bar{\omega}_2{}^2 = 420\, EI/\rho AL^4 \tag{5}$$

Clearly, the frequencies of Eq. (5) are all greater than the frequencies of Eq. (4); therefore, the series expansion of Eq. (1) will converge for each frequency of Eq. (4) and the reduction method has been properly used.

To recover the eliminated degrees of freedom, the back-transformation approximation, Eq. (3), is used and the resulting modal vectors are shown in Table 1 along with the modal

**Table 1 Modal vectors for example problem of Fig. 1**

| Degree of freedom | Present analysis first mode | Present analysis second mode | Guyan method first mode | Guyan method second mode | Exact (Ref. 4) first mode | Exact (Ref. 4) second mode |
|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 0 | 1 | 0 |
| 2 | 0 | 1 | 0 | 1 | 0 | 1 |
| 3 | $1.5706/L$ | $-1.0742$ | $1.50/L$ | $-0.50$ | $1.5708/L$ | $-1$ |
| 4 | $-1.5706/L$ | $-1.0742$ | $-1.50/L$ | $-0.50$ | $-1.5708/L$ | $-1$ |

vectors obtained from the Guyan method[3] and the exact method of Ref. 4. As can be seen from Table 1, the back-transformation of Eq. (3) produces excellent results: an error of $-0.013\%$ for the first mode and an error of $+7.42\%$ for the second mode. Without including the inertia terms in Eq. (3), which is the Guyan back-transformation, the errors are $-4.5\%$ and $-50\%$ for the first and second modes, respectively.

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# Comment on "On Multiple-Shaker Resonance Testing"

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THE characteristic phase-lag method attributed by the author[1] to Bishop and Gladwell was first published in 1948[2] and influenced most of Traill-Nash's early work. Its application to multiple-shaker resonance testing was the subject of an AGARD report[3] in which a principle of minimum reactive energy and the essentially equivalent complex admittance rule is demonstrated. The main result, however, is that, for a limited number of shakers and small enough damping, the forcing amplitude ratios should be such that phase resonance occurs at each shaking point.

The GRAMPA (and later MAMMA) multiple-shaker installation, described in the author's Ref. 6, was based on this result. The performance obtained at Royal Aircraft Establishment with this apparatus was so convincing that it would be interesting to compare this method of regulating the shaker amplitudes with the various approaches described in the article. It may also be mentioned that the characteristic phase-lag and damping theories, duly quoted in many books and articles on the subject of forced response of linear damped systems, were later enlarged and the subject of further publications.[4-6]

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# Reply by Authors to B. Fraeijs de Veubeke

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THE authors express their appreciation to Fraeijs de Veubeke for supplying references to his own work on resonance testing. It was not the authors' intention to present a complete history of the development of resonance test methods. The Bishop and Gladwell reference[1] was quoted because it provides this background as well as an extensive list of references, including references to Fraeijs de Veubeke's 1948[2] and 1956[3] works on the "characteristic phase lag theory."

The technique for determining excitation frequency and force amplitudes in a multiple-shaker test, referred to (perhaps erroneously) by the authors as the "Asher method," had been discussed by Fraeijs de Veubeke, Asher,[4] and others. However, the authors sought, through use of a simulation study, to emphasize the fact that an insufficient number of shakers or an inappropriate group of shakers could produce spurious natural frequencies and unacceptable mode shapes, and to explore more

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